

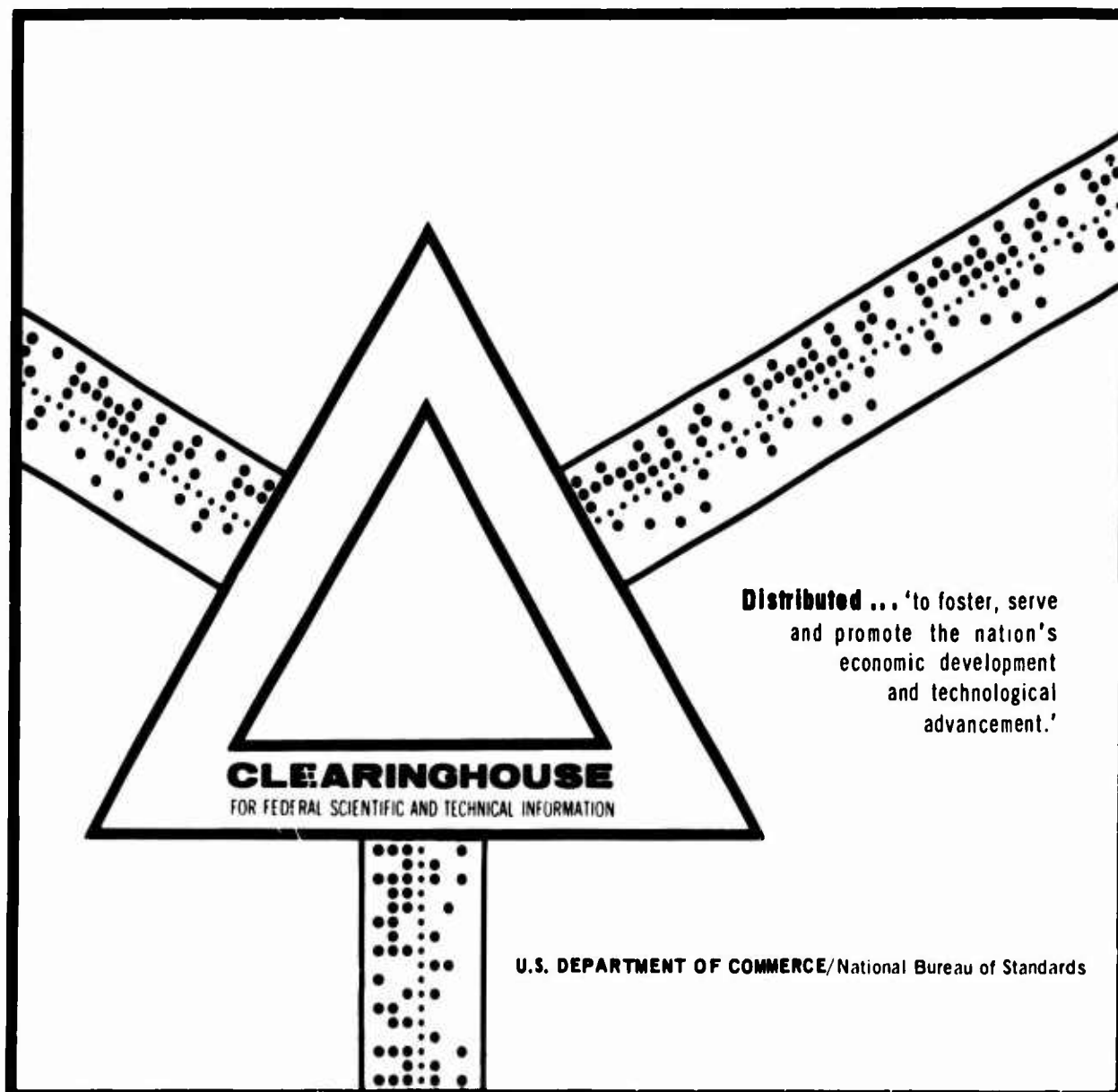
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THE MAXIMUM AND MINIMUM OF A POSITIVE DEFINITE
QUADRATIC POLYNOMIAL ON A SPHERE ARE CONVEX
FUNCTIONS OF THE RADIUS

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July 1969



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CONVEX FUNCTIONS OF THE RADIUS

BY

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COMPUTER SCIENCE DEPARTMENT
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STANFORD UNIVERSITY



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Abstract

It is proved that in euclidean n -space the maximum $M(\rho)$ and minimum $m(\rho)$ of a fixed positive definite quadratic polynomial Q on spheres with fixed center are both convex functions of the radius ρ of the sphere. In the proof, which uses elementary calculus and a result of Forsythe and Golub, $m''(\rho)$ and $M'(\rho)$ are shown to exist and lie in the interval $[2\lambda_1, 2\lambda_n]$, where λ_i are the eigenvalues of the quadratic form of Q . Hence $m''(\rho) > 0$ and $M'(\rho) > 0$.

Summary

Let A be a given symmetric, nonsingular matrix of real elements and order n . Let b be a given column vector of n real elements. For each real column n -vector x , the nonhomogeneous quadratic polynomial

$$Q(x) = (x-b)^T A(x-b)$$

(T denotes transpose) is a real number. Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be the (necessarily) real eigenvalues of A . Let $m(\rho)$ be the minimum of $Q(x)$ on the sphere $S_\rho = \{x: x^T x = \rho^2\}$, and let $M(\rho)$ be the maximum of $Q(x)$ on S_ρ . M. J. D. Powell asked the author whether $m(\rho)$ is a convex function of ρ when A is positive definite. An affirmative answer is given by the theorem:

- (1) Theorem. If A is positive definite (i.e., if $0 < \lambda_1$), then both $m(\rho)$ and $M(\rho)$ are convex functions of ρ , for all $\rho > 0$.

Theorem (1) will follow from the following result:

- (2) Theorem. Let A be any nonsingular matrix. Then for $\rho > 0$, the second derivatives $m''(\rho)$ and $M''(\rho)$ both exist, and

$$(3) \quad m''(\rho) \geq 2\lambda_1 \quad \text{and} \quad M''(\rho) \geq 2\lambda_1.$$

Equality occurs in (3) if and only if $Ab = \lambda_1 b$. Moreover,

$$(4) \quad m''(\rho) \leq 2\lambda_n \quad \text{and} \quad M''(\rho) \leq 2\lambda_n$$

and equality occurs in (4) if and only if $Ab = \lambda_n b$.

Review of Previous Work

The proof of Theorem (2) is based on techniques developed in Forsythe and Golub [1], which dealt only with the case $\rho = 1$. The relevant results of [1] are now summarized and extended to general ρ .

Let $\{u_1, \dots, u_n\}$ be an orthonormal real set of eigenvectors of A , with $Au_i = \lambda_i u_i$ ($i = 1, \dots, n$). Let $b = \sum b_i u_i$. For any vector x in S_ρ at which $Q(x)$ is stationary with respect to S_ρ , there is a real number λ with

$$(5) \quad A(x-b) = \lambda x$$

$$(6) \quad x^T x = \rho^2.$$

Letting $x = \sum x_i u_i$, we find from (5) that

$$(7) \quad x_i = \frac{x_i b_i}{\lambda_i - \lambda},$$

so that (6) becomes

$$(8) \quad r(\lambda) \equiv \sum_{i=1}^n \frac{\lambda_i^2 b_i^2}{(\lambda_i - \lambda)^2} = \rho^2.$$

For each given value of $\rho > 0$, equation (8) determines from 2 to $2n$ real values of λ . For each λ so determined, equation (5) determines one or more vectors x^λ (if all $b_i \neq 0$, then x^λ is unique). For any x^λ , we have

$$(9) \quad Q(x^\lambda) = r(\lambda),$$

where

$$(10) \quad r(\lambda) = \lambda^2 \sum_{i=1}^n \frac{\lambda_i b_i^2}{(\lambda_i - \lambda)^2} .$$

Now $Q(x)$ is stationary with respect to S_ρ at any x^λ . For given ρ , let $\Lambda_L = \Lambda_L(\rho)$ and $\Lambda_R = \Lambda_R(\rho)$ be the smallest resp. largest values of λ satisfying equation (8). Theorem (4.1) of [1] states that $f(\Lambda_L)$ and $f(\Lambda_R)$ are the minimum resp. maximum values of $Q(x)$ on S_ρ .

Much of [1] was devoted to the singular cases where some $b_i = 0$. For the present investigation, where we are interested only in the values of $Q(x)$, we simply omit from the sums (8) and (10) all terms with $b_i = 0$, and reduce n , if necessary. Having done that, it is then clear from (8) that, for any ρ ,

$$(11) \quad \Lambda_L < \lambda_1 \quad \text{and} \quad \lambda_n < \Lambda_R .$$

This concludes the necessary summary of [1].

As a digression, the author notes that the main theorems (2.7) and (4.1) of [1] were proved in [1] by studying $f(\lambda)$ and $g(\lambda)$ for complex values of λ . In late 1965, Professor W. Kahan [unpublished] showed us how to prove those theorems more simply, using only real values of λ .

Proof of Theorem (2).

With the above apparatus our problem is reduced to an exercise in the differential calculus. For each $\rho > 0$ we determine a unique Lagrange multiplier $\lambda = \lambda(\rho)$ from (8) -- either the minimal Λ_L or maximal Λ_R . For ease of exposition, suppose $\lambda(\rho) = \Lambda_L$. Then the function

$$(12) \quad m(\rho) = f(\lambda(\rho))$$

is determined from (10). Since $f(\lambda)$ and $g(\lambda)$ are analytic for $\lambda < \lambda_1$, the function $m(\rho)$ has derivatives of all order. We shall determine $m''(\rho)$ by calculus. To simplify some expressions, we introduce the abbreviations

$$(13) \quad \alpha_p = \sum_{i=1}^n \frac{\lambda_i^2 b_i^2}{(\lambda_i - \lambda)^2} \quad (p = 2, 3, 4) .$$

Differentiating (10) and simplifying, we find:

$$(14) \quad \frac{df}{d\lambda} = 2\lambda\alpha_3 ;$$

$$(15) \quad \frac{d^2f}{d\lambda^2} = 2\alpha_3 + 6\lambda\alpha_4 .$$

Now equation (8) states that, when $\lambda = \lambda(\rho)$,

$$(16) \quad \alpha_2 = \rho^2 .$$

Differentiating (8) twice with respect to ρ yields

$$(17) \quad \frac{d\lambda}{d\rho} \alpha_3 = \rho ;$$

$$(18) \quad \frac{d^2\lambda}{d\rho^2} \alpha_3 + 3\left(\frac{d\lambda}{d\rho}\right)^2 \alpha_4 = 1 .$$

Solving (17) and (18) in turn, we find

$$(19) \quad \frac{d\lambda}{d\rho} = \frac{\rho}{\alpha_3} ;$$

$$(20) \quad \frac{d^2\lambda}{d\rho^2} = \frac{1}{\alpha_3} - \frac{3\rho^2\alpha_4}{\alpha_3^2} .$$

Now, by the chain rule,

$$\frac{dm}{d\rho} = \frac{df}{d\lambda} \cdot \frac{d\lambda}{d\rho} ,$$

and

$$(21) \quad \frac{d^2 m}{d\rho^2} = \frac{d^2 f}{d\lambda^2} \left(\frac{d\lambda}{d\rho} \right)^2 + \frac{df}{d\lambda} \cdot \frac{d^2 \lambda}{d\rho^2} .$$

We now substitute into (21) the expressions (14), (15), (19), and (20).

We find that

$$(22) \quad m''(\rho) = \frac{d^2 m}{d\rho^2} = (2\alpha_3 + 6\lambda\alpha_4) \frac{\rho^2}{\alpha_3^2} + 2\lambda\alpha_3 \left(\frac{1}{\alpha_3} - \frac{3\rho^2\alpha_4}{\alpha_3^3} \right) .$$

Hence

$$\frac{1}{2} m''(\rho) = \lambda + \frac{\rho^2}{\alpha_3} = \frac{1}{\alpha_3} (\lambda\alpha_3 + \alpha_2) , \quad \text{by (16).}$$

Simplifying,

$$\frac{1}{2} m''(\rho) = \frac{1}{\alpha_3} \sum_{i=1}^n \frac{\lambda_i^3 b_i^2}{(\lambda_i - \lambda)^3} , \quad \text{or}$$

$$(23) \quad \frac{1}{2} m''(\rho) = \sum_{i=1}^n \frac{\lambda_i^3 b_i^2}{(\lambda_i - \lambda)^3} \bigg/ \sum_{i=1}^n \frac{\lambda_i^2 b_i^2}{(\lambda_i - \lambda)^3} .$$

Formula (23) is the end of our calculus exercise. In it, λ is determined from solving (8). Note by (11) that the factors $(\lambda_i - \lambda)^3$ all have the same sign for $i = 1, 2, \dots, n$, whether $\lambda = \lambda_L$ or $\lambda = \lambda_R$. Hence $\frac{1}{2} m''(\rho)$ is a weighted average with positive weights of the $\{\lambda_i\}$.

It follows that $\frac{1}{2} m''(\rho) \geq \lambda_1$, with equality only when all λ_i in (23) are equal to λ_1 , i.e., if $b_i = 0$ for $\lambda_i > \lambda_1$. This proves (3), and (4) is proved analogously. This concludes the proof of Theorem (2).

It would be desirable to have a simple geometrical proof.

What if A is singular?

If A is singular, that is, if some $\lambda_i = 0$, the situation is somewhat more complicated, just as the case where some $\lambda_i b_i = 0$ is complicated in [1]. Theorem (2) fails to hold for semidefinite matrices, because $m''(\rho)$ may not exist for some ρ , as the following example shows:

(24) Example. For $n = 2$ let $Q(x) = (x_2 - 1)^2$, where $x = (x_1, x_2)^T$.

Then

$$m(\rho) = \begin{cases} 1-\rho & , \quad 0 \leq \rho \leq 1 \\ 0 & , \quad 1 \leq \rho < \infty \end{cases}$$

so $m''(1)$ does not exist.

If $\lambda_1 = 0$, the Lagrange multiplier remains at $\lambda = 0$ for all sufficiently large ρ .

Theorem (1) can easily be extended to semidefinite matrices by continuity. We have

(25) Theorem. If A is positive semidefinite (i.e., if $0 \leq \lambda_1$),
then both $m(\rho)$ and $M(\rho)$ are convex functions of ρ for $\rho > 0$.

In proof, we note that $m(\rho)$ and $M(\rho)$ are continuous functions of the elements of A. If A is semidefinite, it can be approximated by a definite matrix A_ϵ , for which m_ϵ and M_ϵ are convex, with $\|A - A_\epsilon\| < \epsilon$. Letting $\epsilon \rightarrow 0$, we find that $m = \lim m_\epsilon$ and $M = \lim M_\epsilon$ are convex.

Reference

- [1] George E. Forsythe and Gene H. Golub, "On the stationary values of a second-degree polynomial on the unit sphere", J. Soc. Indust. Appl. Math., vol. 13 (1965), pp. 1050-1068.

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